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ANALYSIS OF THE BOUNDARY LAYER IN THE AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR A RADIALLY MULTILAYERED CYLINDER AND THE PROPAGATION OF AXISYMMETRIC WAVES†

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The structures of the boundary layer are investigated using the example of the axisymmetric problem of the theory of elasticity for a radially multilayered cylinder with alternating hard and soft layers. On the basis of an asymptotic analysis of the problem, an applied theory of stretching is proposed, which takes into account weakly decaying boundary-layer solutions. The propagation of asymmetric waves in a radially multilayered cylindrical waveguide is investigated. © 1998 Elsevier Science Ltd. All rights reserved.

It was shown in [1, 2] that for multilayered bodies with alternating hard and soft layers, weakly decaying boundary-layer solutions exist which penetrate fairly far into the depth of the region and which provide an important correction to the penetrating solutions. The method used in those papers was extended in [3] to problems of steady torsional oscillations of a radially multilayered cylinder with alternating hard and soft layers.

1. We will consider the axisymmetric problem of the theory of elasticity for a circular radially multilayered cylinder consisting of alternating hard and soft layers with numbers $n = 2r - 1$. We will assume that the inner and outer layers are hard. Each hard layer is given an odd number $j = 1, 3, \ldots, n$, while each soft layer is given an even number $i = 2, 4, \ldots, n - 1$. We will assume, for simplicity, that the elastic properties of all the hard and soft layers are the same: the shear moduli $G_j = G_r$, $G_i = G_s$, Poisson's ratios $\nu_j = \nu_r$, $\nu_i = \nu_s$, and the densities $m_j = m_r$, $m_i = m_s$. Suppose the cylinder occupies a volume $\Gamma = \{r \in [r_0, r_1], \varphi \in [0, 2\pi], z \in [-L, L]\}$. The inner and outer radii of the kth layer will be denoted by r_{0k} and r_{1k}, respectively.

The equations of equilibrium of the kth layer in terms of displacements in dimensionless variables have the form

$$\Delta u_{\rho k} - \frac{1}{\rho^2} u_{\rho k} + \frac{1}{1-2\nu_k} \frac{\partial \theta_k}{\partial \rho} = 0 \qquad (1.1)$$

$$\Delta u_{\zeta k} + \frac{1}{1-2\nu_k} \frac{\partial \theta_k}{\partial \zeta} = 0$$

Here

$$\rho = \frac{r}{R_0}, \quad \zeta = \frac{z}{R_0}, \quad \rho \in [\rho_{01}, \rho_{1n}]$$

$$\zeta \in [-l, l], \quad l = \frac{L}{R_0}, \quad u_{\rho k} = \frac{u_{rk}}{R_0}, \quad u_{\zeta k} = \frac{u_{zk}}{R_0}$$

$$R_0 = \frac{r_{01} + r_{1n}}{2}, \quad \theta_k = \frac{\partial u_{\rho k}}{\partial \rho} + \frac{u_{\rho k}}{\rho} + \frac{\partial u_{\zeta k}}{\partial \zeta}$$

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$$\Delta = \Delta_1 + \frac{\partial^2}{\partial \zeta^2}, \quad \Delta_1 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}, \quad k = 1,2,\ldots,n$$

We will assume that the side surface of the cylinder is stress-free, i.e.

$$\begin{aligned} &\sigma_{\rho\rho}^{(1)}(\rho_{01},\zeta) = \sigma_{\rho\zeta}^{(1)}(\rho_{01},\zeta) = 0 \\ &\sigma_{\rho\rho}^{(n)}(\rho_{1n},\zeta) = \sigma_{\rho\zeta}^{(n)}(\rho_{1n},\zeta) = 0 \end{aligned} \tag{1.2}$$

We will assume the connection between the layers to be rigid, which means that the following matching conditions are satisfied

$$\boldsymbol{\sigma}_m(\rho_{1m},\zeta) = \boldsymbol{\sigma}_m(\rho_{0,m+1},\zeta), \mathbf{u}_m(\rho_{1m},\zeta) = \mathbf{u}_{m+1}(\rho_{0,m+1},\zeta) \tag{1.3}$$

$$\boldsymbol{\sigma}_m = (\sigma_{\rho\rho}^{(m)}, \sigma_{\rho\zeta}^{(m)}), \quad \mathbf{u}_m = (u_{\rho m}, u_{\zeta m}); \quad m = 1,2,\ldots,n-1$$

We will assume that arbitrary boundary conditions, which maintain the cylinder in equilibrium, are specified on the ends of the cylinder.

We will seek the solution of problems (1.1)–(1.3) in the form [4]

$$u_{\rho k} = u_k(\rho)\, a'(\zeta), \quad u_{\zeta k} = w_k(\rho)\, a(\zeta) \tag{1.4}$$

where the function $a(\zeta)$ is subject to the condition

$$a''(\zeta) - \alpha^2 a(\zeta) = 0$$

where α is a certain parameter.

Substituting (1.4) into (1.1)–(1.3) we obtain the following non-self-conjugate eigenvalue problems

$$\begin{aligned} &(L_{0k} + L_{1k} + \alpha^2(L_{2k} + L_{3k} - L_{1k}))\,\mathbf{v}_k = \mathbf{0} \\ &M_1(\alpha^2)\,\mathbf{v}_1(\rho_{01}) = M_n(\alpha^2)\,\mathbf{v}_n(\rho_{1n}) = \mathbf{0} \\ &M_m(\alpha^2)\,\mathbf{v}_m(\rho_{1m}) = M_{m+1}(\alpha^2)\,\mathbf{v}_{m+1}(\rho_{0,m+1}) \\ &\mathbf{v}_m(\rho_{1m}) = \mathbf{v}_{m+1}(\rho_{0,m+1}) \end{aligned} \tag{1.5}$$

where $\mathbf{v}_k = (u_k, w_k)$ and L_{tk} and M_k are matrix differential operators of the form

$$L_{0k} = \begin{Vmatrix} \Delta_1 - \dfrac{1}{\rho^2} & 0 \\ 0 & \Delta_1 \end{Vmatrix}, \quad L_{1k} = \begin{Vmatrix} 0 & \dfrac{1}{2(1-\nu_k)}\dfrac{d}{d\rho} \\ 0 & 0 \end{Vmatrix}$$

$$L_{2k} = \begin{Vmatrix} 0 & \dfrac{1}{2(1-\nu_k)}\dfrac{d}{d\rho} \\ \dfrac{1}{1-2\nu_k}\left(\dfrac{d}{d\rho} + \dfrac{1}{\rho}\right) & 0 \end{Vmatrix}$$

$$L_{3k} = \begin{Vmatrix} \dfrac{1-2\nu_k}{2(1-\nu_k)} & 0 \\ 0 & \dfrac{2(1-\nu_k)}{1-2\nu_k} \end{Vmatrix}$$

$$M_k(\alpha^2) \equiv M_{0k} + M_{1k} + \alpha^2(M_{2k} - M_{1k})$$

$$M_{0k} = \begin{Vmatrix} \dfrac{G_k}{1-2\nu_k}\left((1-\nu_k)\dfrac{d}{d\rho} + \dfrac{\nu_k}{\rho}\right) & 0 \\ 0 & G_k\dfrac{d}{d\rho} \end{Vmatrix}$$

$$M_{1k}=\begin{Vmatrix}0 & \dfrac{G_k\nu_k}{1-2\nu_k}\\ 0 & 0\end{Vmatrix},\quad M_{2k}=\begin{Vmatrix}0 & \dfrac{G_k\nu_k}{1-2\nu_k}\\ G_k & 0\end{Vmatrix}$$

2. We will introduce the small parameter $p = G_s/G_r$ as a characteristic of the relative stiffness of the layers and we will investigate eigenvalue problem (1.5) as $p \to 0$.

Eigenvalue problem (1.5) reduces to investigating a certain homogeneous algebraic system with a matrix whose elements depend analytically on the eigenvalue parameter α and linearly on the parameter p. Applying the theory of perturbations of linear operators [5] to this algebraic system, we conclude that the following theorem holds.

Theorem. The spectrum $\Lambda(p)$ of problem (1.5) as $p \to 0$ can be represented in the form

$$\Lambda(p)=\Lambda_0(p)\cup\Lambda_-(p)\cup\Lambda_+^{(1)}(p)\cup\Lambda_+^{(2)}(p)$$

where

1. $\Lambda_0(p)$ consists of the double eigenvalue $\alpha_0 = 0$;
2. $\Lambda_-(p)$ consists of $2(r-1)$ real eigenvalues of the form

$$\alpha_t = p^{1/2}\eta_t + O(p^{3/2}) \tag{2.1}$$

were η_t is a non-zero eigenvalue of the homogeneous Jacobi algebraic system

$$C\mathbf{X}-\eta^2 B\mathbf{X}=0 \tag{2.2}$$

$$\mathbf{X}=(X_1,X_3,\ldots,X_n),\quad B=\operatorname{diag}\|b_{jj}\|,\quad b_{jj}=(1+\nu_r)(\rho_{1j}^2-\rho_{0j}^2)$$

$$C=\begin{Vmatrix}c_1 & -c_1 & 0 & \cdots & 0 & 0 & 0\\ -c_1 & c_1+c_3 & -c_3 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & -c_{n-2} & c_{n-2}\end{Vmatrix},\quad c_j=\left(\ln\frac{\rho_{0,j+2}}{\rho_{1j}}\right)^{-1}$$

3. $\Lambda_+^{(1)}(p)$ consists of r sets of eigenvalues of the form

$$\alpha_{tj}^{(1)}=\alpha_{tj}+O(p^{\delta}) \tag{2.3}$$

where α_{tj} is the root of the equation

$$\begin{aligned}&\alpha^4\rho_{0j}^2\rho_{1j}^2E_{j00}^2+\alpha^2\rho_{0j}^2f_{1j}(\alpha)\,E_{j01}^2+\alpha^2\rho_{1j}^2f_{0j}(\alpha)\,E_{j10}^2+\\&+f_{0j}(\alpha)\,f_{1j}(\alpha)\,E_{j11}^2-4\pi^{-2}(f_{0j}(\alpha)+f_{1j}(\alpha))=0\\&E_{k\beta\gamma}=J_\beta(\alpha\rho_{0k})\,Y_\gamma(\alpha\rho_{1k})-J_\gamma(\alpha\rho_{1k})\,Y_\beta(\alpha\rho_{0k})\\&f_{\beta j}(\alpha)=\alpha^2\rho_{\beta j}^2+2(\nu_r-1);\quad \beta,\gamma=0,1\end{aligned} \tag{2.4}$$

4. $\Lambda_+^{(2)}(p)$ consist of $r-1$ sets of eigenvalues of the form

$$\alpha_{ti}^{(2)}=\alpha_{ti}+O(p^{\delta}) \tag{2.5}$$

where α_{ti} is the root of the equation

$$\begin{aligned}&\alpha^2\rho_{0i}^2\rho_{1i}^2(E_{i11}^2+E_{i00}^2+E_{i10}^2+E_{i01}^2)-4\alpha(1-\nu_s)\,\rho_{0i}\rho_{1i}^2(E_{i00}E_{i10}+E_{i11}E_{i01})-\\&-4\alpha(1-\nu_s)\,\rho_{0i}^2\rho_{1i}(E_{i11}E_{i10}+E_{i00}E_{i01})+\\&+16(1-\nu_s)^2\rho_{0i}\rho_{1i}E_{i00}E_{i11}-4\pi^{-2}(\rho_{0i}^2+\rho_{1i}^2)=0\end{aligned} \tag{2.6}$$

Here J_β, Y_β are Bessel functions of the first and second kind respectively and $\delta = 1$ if none of the roots of Eqs (2.4) and (2.6) is the same, or if there are some roots that are the same but with the condition that $i \neq j+1$, $i \neq j-1$, and $\delta = 1/2$ if when $i = j+1$ or $i = j-1$ the corresponding equations have at least a pair of similar roots.

The double eigenvalues $\alpha_0 = 0$ define Saint-Venant solutions, which penetrate without attenuation inside the cylinder. These solutions have the form

$$u_{\rho k}^{(0)} = D_0(-\nu_r \rho X_0 + O(p)), \quad u_{\zeta k}^{(0)} = X_0(E_0 + D_0\zeta)$$
$$X_0 = (b_{11} + b_{33} + \ldots + b_{nn})^{-1/2} \tag{2.7}$$

(E_0 and D_0 are arbitrary constants).

The boundary-layer solutions are defined as the sum of elementary solutions of the form

$$u_{\rho k}^{(t)} = u_{kt}(\rho)\, a_t'(\zeta), \quad u_{\zeta k}^{(t)} = w_{kt}(\rho)\, a_t(\zeta)$$
$$a_t(\zeta) = D_t \exp(\alpha_t\zeta) + E_t \exp(-\alpha_t\zeta)$$

Here D_t and E_t are arbitrary constants, α_t, (u_{kt}, w_{kt}) are eigenvalues and eigenfunctions of problem (1.5) and $\operatorname{Re}\alpha_t > 0$.

It follows from the theorem that in the case of a radially multilayered cylinder with alternating hard and soft layers there is a finite number of eigenvalues belonging to $\Lambda_-(p)$, which, for small p, tend to zero. This means that the elementary solutions corresponding to the lower part of the spectrum of $\Lambda_-(p)$ as $p \to 0$ can penetrate fairly deeply far from the ends and provide a considerable correction to the penetrating solution. This leads to a breakdown of the Saint–Venant principle in its classical formulation. The set of these solutions will be called a weak boundary layer. The eigenvalues belonging to $\Lambda_+^{(1)}(p)$, $\Lambda_+^{(2)}(p)$ as $p \to 0$ have finite limits. The elementary solutions corresponding to the upper part of the spectrum of $\Lambda_+^{(1)}(p)$ and $\Lambda_+^{(2)}(p)$ decay strongly with distance from the ends. This set of solutions will be called a strong boundary layer.

Note that for small p the stratification of the spectrum into "lower" and "higher" parts will be more pronounced the greater the difference $\inf_{t,k}\alpha_{tk} - \sup_{tk} p^{1/2}\eta_t$.

The distribution of the displacements over the radii of the corresponding weak boundary layer in the elementary solutions can be represented in terms of the eigenfunctions $\mathbf{X}_t = (X_{1t}, X_{3t}, \ldots, X_{nt})$ of algebraic system (2.2) as follows:

$$u_{kt} = F_t(u_{kt0} + O(p)), \quad w_{kt} = F_t(w_{kt0} + O(p))$$
$$u_{jt0} = -\nu_r \rho X_{jt}, \quad w_{jt0} = X_{jt} \tag{2.8}$$
$$u_{it0} = (4\omega_i(\nu_s - 1)(\tau_{0i} - \tau_{1i}))^{-1}[((4(\nu_s - 1)\nu_r + 1)\times$$
$$\times \rho_{0i}^2 \rho^{-1}(\rho^2 - \rho_{1i}^2)(\tau_{0i} - \tau_{1i}) - \rho\omega_i\tau_{1i}X_{i-1,t}) -$$
$$-((4(\nu_s - 1)\nu_r + 1)\rho_{1i}^2\rho^{-1}(\rho^2 - \rho_{0i}^2)(\tau_{0i} - \tau_{1i}) - \rho\omega_i\tau_{0i})X_{i+1,t}]$$
$$w_{it0} = (\tau_{0i} - \tau_{1i})^{-1}(\tau_{0i}X_{i+1,t} - \tau_{1i}X_{i-1,t})$$
$$\tau_{0i} = \ln\frac{\rho}{\rho_{0i}}, \quad \tau_{1i} = \ln\frac{\rho}{\rho_{1i}}, \quad \omega_i = \rho_{1i}^2 - \rho_{0i}^2$$

Here F_t is a normalizing factor.

3. It was shown in [1, 2] that the lower part of the spectrum corresponds to a certain applied theory which, in addition to the penetrating solutions, includes all weak boundary layers.

The penetrating solutions with a weak boundary layer can be given the following mechanical interpretation. We will assume that the stress–strain state of the hard layers corresponds to pure stretching along the axis of symmetry, while the soft layers possess Winkler shear properties. These hypotheses enable the displacements in the hard and soft layers to be represented in the form

$$u_{\rho j} = -\nu_r \rho g_j'(\zeta), \quad u_{\zeta j} = g_j(\zeta)$$

$$u_{\rho i} = 0, \quad u_{\zeta i} = \tau_{0i}^{-1}(\tau_{0i} g_{i+1}(\zeta) - \tau_{1i} g_{i-1}(\zeta)) \tag{3.1}$$

The stress–strain state in each of the hard and soft layers will then be as follows:

$$\sigma_{\zeta\zeta}^{(j)} = 2G_r(1+\nu_r)\, g_j(\zeta)$$

$$\sigma_{\rho\zeta}^{(i)} = G_s(\rho(\tau_{0i} - \tau_{1i}))^{-1}(g_{i+1}(\zeta) - g_{i-1}(\zeta)) \tag{3.2}$$

The remaining components of the stress tensors are zero. In order to obtain the boundary-value problem corresponding to the chosen model of the stress–strain state, we will use the Lagrange variational principle

$$\delta\Pi - \delta A = 0 \tag{3.3}$$

where δA is the variation of the work done by the external forces and $\delta\Pi$ is the variation of the deformation energy.

To determine δA we will use, for example, the fact that the following boundary conditions are specified on the ends

$$u_{\zeta k}(\rho, -l) = 0, \quad \sigma_{\zeta\zeta}^{(k)}(l) = \sigma_{0k}$$

$$\sigma_{0k} = \sigma_{0j}, \quad \rho \in [\rho_{0j}, \rho_{1j}]; \quad \sigma_{0k} = 0, \quad \rho \in [\rho_{0i}, \rho_{1i}] \tag{3.4}$$

Assuming δg_j to be independent variations, we obtain the following boundary-value problem from the variational equation (3.3) taking (3.1), (3.2) and (3.4) into account

$$-B\mathbf{g}'' + pC\mathbf{g} = 0 \tag{3.5}$$

$$\mathbf{g}(-l) = 0, \quad B\mathbf{g}'(l) = \mathbf{d} \tag{3.6}$$

where

$$\mathbf{g} = (g_1, g_3, \ldots, g_n), \quad \mathbf{d} = (d_1, d_3, \ldots, d_n), \quad d_j = \omega_j \sigma_{0j} / (2G_r)$$

If the solution of Eq. (3.5) is sought in the form

$$\mathbf{g} = \mathbf{X}a(\zeta), \quad \alpha = \rho^{1/2}\eta$$

we arrive at problem (3.2).

In view of the fact that problem (2.2) is self-conjugate, the eigenvectors $\mathbf{X}_t = (X_{1t}, X_{3t}, \ldots, X_{nt})$ corresponding to the eigenvalue $\lambda_t = \eta_t^2$ may be subject to the condition

$$(B\mathbf{X}_t, \mathbf{X}_k) = \sum_{j=1,3,\ldots}^{n} b_{jj} X_{jt} X_{jk} = \delta_{tk} \tag{3.7}$$

where $\lambda_0 = 0$ is the eigenvalue vector and the eigenvector $\mathbf{X}_0 = (X_0, \ldots, X_0)$ corresponds to it, where X_0 is defined by the last formula of (2.1).

The general solution of Eq. (3.5) can be represented in the form

$$\mathbf{g}(\xi) = \mathbf{X}_0(E_0 + D_0\xi) + \sum_{t=1}^{r-1} \mathbf{X}_t (E_t \exp(-\alpha_t \xi) + D_t \exp(\alpha_t \xi))$$

The constants E_0, D_0, E_t and D_t can be found from (3.6), taking (3.7) into account, as follows:

$$D_0 = X_0 \sum_{j=1,3,\ldots}^{n} d_j, \quad E_0 = lD_0$$

$$D_t = (2\alpha_t \operatorname{ch}(2\alpha_t l))^{-1} \exp(\alpha_t l) \sum_{j=1,3,\ldots}^{n} d_j X_{jt}, \quad E_t = -\exp(-2\alpha_t l) D_t$$

It can be seen from (3.1) that the displacement field corresponding to the particular solution $g_0 = X_0(E_0 + D_0\xi)$ is equivalent to the Saint-Venant solution. The solution corresponding to the non-zero eigenvalues η_t is the first approximation in p of the weak boundary layer.

4. As an example we will consider a three-layer cylinder. In this case

$$\eta_1^2 = c_1(b_{11}^{-1} + b_{33}^{-1})$$

$$\mathbf{X}_1 = (X_{11}, X_{13}), \quad X_{11} = \left(\frac{b_{33}}{b_{11}}\right)^{1/2} X_0, \quad X_{13} = -\left(\frac{b_{11}}{b_{33}}\right)^{1/2} X_0, \quad X_0 = (b_{11} + b_{33})^{-1/2}$$

For a three-layer cylinder with parameters: (1) $\rho_{01} = 0.3$; $\rho_{11} = 0.4$; $\rho_{03} = 0.8$; $\rho_{13} = 1$; $\nu_r = 0.25$ and (2) $\rho_{01} = 0.8$; $\rho_{11} = 0.85$; $\rho_{03} = 0.95$; $\rho_{13} = 1$; $\nu_r = 0.25$ we obtain correspondingly $\eta_1 = 4.4$ and $\eta_1 = 12.7$.

In Fig. 1, for a three-layer cylinder, we show curves of the attenuation factor of the weak boundary-layer solution as a function of the parameter $-\lg p$ in cases 1 and 2, when $p \in [10^{-4}, 10^{-1}]$. It can be seen that as the stiffness of the filler increases the depth of penetration of the weak boundary layer is reduced. Also, for a cylinder with a fairly soft filler the weak boundary-layer solutions penetrate fairly far into the depth of the region. The depth of penetration of the weak boundary layer increases as the cylinder thickness increases.

For a single-layer cylinder with an inner radius $\rho_0 = 0.8$ and an outer radius $\rho_1 = 1$ the attenuation factors of the boundary-layer solutions are 2.8.

5. Consider the propagation of steady axisymmetric elastic waves in a radially multilayered cylindrical waveguide, consisting of alternating hard and soft layers with number $n = 2r - 1$.

Assuming that the side surfaces are stress-free and the connection between the layers is rigid, substituting into the equations of motion

$$\mathbf{u}_k(\rho, \xi, t) = \mathbf{v}_k(\rho)\exp(i(\alpha\xi - \omega t))$$

we obtain the eigenvalue problems

$$\begin{aligned} &(L_{0k} + i\alpha L_{2k} - \alpha^2 L_{3k} + \Omega^2 L_{4k})\mathbf{v}_k = 0 \\ &N_1(\alpha)\mathbf{v}_1(\rho_{01}) = N_n(\alpha)\mathbf{v}_n(\rho_{1n}) = 0 \\ &N_m(\alpha)\mathbf{v}_m(\rho_m) = N_{m+1}(\alpha)\mathbf{v}_{m+1}(\rho_{0,m+1}) \\ &\mathbf{v}_m(\rho_{1m}) = \mathbf{v}_{m+1}(\rho_{0,m+1}) \end{aligned} \qquad (5.1)$$

where L_{4k} and N_k are matrix operators of the form

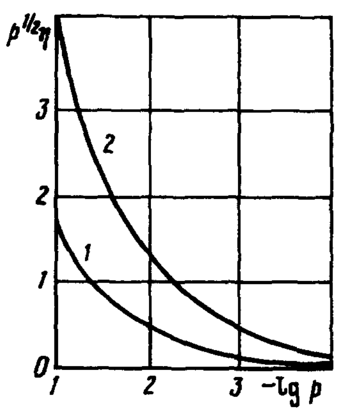

Fig. 1.

$$L_{4k} = \begin{Vmatrix} M_k^{-2} & 0 \\ 0 & e_k^{-2} \end{Vmatrix}, \quad N_k(\alpha) = M_{0k} + i\alpha M_{2k}$$

$$\Omega^2 = \frac{m_0\omega^2 R_0^2}{G_0}, \quad e_k = \left(\frac{G_k}{m_k}\right)^{1/2}, \quad \mu_k = \left(\frac{2(1-\nu_k)}{1-2\nu_k}\right)^{1/2} e_k$$

$$G_k = \frac{G_k}{G_0}, \quad m_k = \frac{m_k}{m_0} -$$

is the dimensionless angular frequency, ω is the frequency of the oscillation and G_0 and m_0 are certain characteristic parameters which have the dimensions of shear modulus and density.

6. The determination of the wave pattern in a radially multilayered cylindrical waveguide involves constructing the dispersion curves $\alpha_s = \alpha_s(\Omega)$ defined by the eigenvalue set (5.1).

We know that homogeneous waves, propagating along the axis of the cylinder and transferring energy, correspond to real eigenvalues α_s of eigenvalue problem (5.1). Imaginary and complex eigenvalues α_s of problem (5.1) define inhomogeneous waves which do not transfer energy.

We will now investigate the real dispersion sets $\alpha_s(\Omega)$ since, for unbounded bodies, the main characteristics of these waveguides are real dispersion curves $\alpha_s = \alpha_s(\Omega)$.

Problem (5.1) with $\alpha = 0$ describes thickness resonances. In the (α, Ω) plane each curve intersects the frequency axis at the point $(0, \Omega_s)$, where Ω_s are the frequencies of a thickness resonance, and is the origin of the dispersion curves.

We will investigate the family of thickness resonance frequencies. To do this we will consider the limit problem, putting $\alpha = 0$. In this case (5.1) splits into two independent problems

$$\text{(a)}\ u_{\rho k} \equiv 0, \ u_{\xi k} = w_k(\rho)\ (\sigma_{\rho\rho}^{(k)} = \sigma_{\varphi\varphi}^{(k)} = \sigma_{\xi\xi}^{(k)} \equiv 0)$$
$$\text{(b)}\ u_{\rho k} = u_k(\rho), \ u_{\xi k} \equiv 0\ \ (\sigma_{\rho\xi}^{(k)} \equiv 0)$$

The first problem corresponds to longitudinal shear oscillations while the second corresponds to purely radial oscillations [7].

Theorem. $\Lambda_1(p)$, $\Lambda_2(p)$—the eigenvalues of the problems corresponding to longitudinal shear and purely radial oscillations as $p \to 0$, are a denumerable set and can be represented in the form

$$\Lambda_1(p) = \Lambda_{10}(p) \cup \Lambda_{11}(p)\Lambda_{12}(p)$$

$$\Lambda_2(p) = \Lambda_{21}(p) \cup \Lambda_{22}(p)$$

where

1. $\Lambda_{10}(p)$ is a set consisting of the double eigenvalue $\Omega = 0$ and $2(r-1)$ eigenvalues of the form

$$\Omega_s = p^{1/2}\gamma_s + O(p^{3/2})$$

where γ_s^2 are non-zero eigenvalues of the homogeneous Jacobi algebraic system

$$C\mathbf{X} - \gamma^2 D\mathbf{X} = 0 \tag{6.1}$$

$D = \operatorname{diag}\|d_{jj}\|$, $d_{jj} = (\rho_{1j}^2 - \rho_{0j}^2)/(2l_j^2)$ and C is a matrix, the same as in (2.2);

2. $\Lambda_{11}(p)$ is a set consisting of r sets of eigenvalues of the form

$$\Omega_{tj} = \Omega_{0tj}^{(1)} + O(p^{\delta_1})$$

where $\Omega_{0tj}^{(1)}$ is the root of the equation

$$E_{j11}(\varepsilon_{1j}) = 0 \tag{6.2}$$

3. $\Lambda_{12}(p)$ consists of $r-1$ sets of eigenvalues of the form

$$\Omega_{ti} = \Omega_{0ti}^{(1)} + O(p^{\delta_1})$$

where $\Omega_{0ti}^{(1)}$ is the root of the equation

$$E_{i00}(\varepsilon_{1i}) = 0 \tag{6.3}$$

4. $\Lambda_{21}(p)$ consists of r sets of eigenvalues of the form

$$\Omega_{tj} = \Omega_{0tj}^{(2)} + O(p^{\delta_2})$$

where $\Omega_{0tj}^{(2)}$ is the root of the equation

$$\mu_j^2\Omega^2\rho_{0j}\rho_{1j}E_{j00}(\varepsilon_{2j}) - 2\mu_j e_j^2\Omega(\rho_{0j}E_{j01}(\varepsilon_{2j}) + \rho_{1j}E_{j10}(\varepsilon_{2j})) + 4e_j^4 E_{j11}(\varepsilon_{2j}) = 0 \tag{6.4}$$

5. $\Lambda_{22}(p)$ consists of $r-1$ sets of eigenvalues of the form

$$\Omega_{ti} = \Omega_{0ti}^{(2)} + O(p^{\delta_2})$$

where $\Omega_{0ti}^{(2)}$ is the root of the equation

$$E_{i11}(\varepsilon_{2i}) = 0 \tag{6.5}$$

Here $\varepsilon_{1k} = \Omega/e_k$, $\varepsilon_{2k} = \Omega/\mu_k$; $\delta_1 = 1(\delta_2 = 1)$ if there are no roots that are the same among the roots of (6.2) and (6.3) ((6.4) and (6.5)), but with the condition that $i \neq j+1$, $i \neq j-1$, and $\delta_1 = 1/2$ ($\delta_2 = 1/2$), if for $i = j-1$ or $i = j+1$ the corresponding equations have at least a pair of similar roots.

The eigenfunctions corresponding to $\Lambda_{10}(p)$ have the form

$$w_{kt} = w_{kt0}(\rho) + O(p) \tag{6.6}$$

The function $w_{kt0}(\rho)$ is given by formula (2.8).

We will consider the construction of asymptotic approximations in the neighbourhood of $(\alpha, \Omega) = (0, 0)$. When $\Omega = 0$ the point $\alpha = 0$ is a double eigenvalue of (5.1). Using methods of branching theory [8] in the neighbourhood of the point $(\alpha, \Omega) = (0, 0)$, the solution of eigenvalue problem (5.1) will be sought in the form

$$\alpha = q_1\Omega + q_2\Omega^2 + \dots, \quad \mathbf{v}_k = \mathbf{v}_{0k} + \Omega\mathbf{v}_{1k} + \dots \tag{6.7}$$

Substituting (6.6) into (5.1) we obtain a certain recurrent system, after integrating which we obtain

$$q_1^{\pm} = \pm\left(\frac{\sum_{k=1}^{n} m_k(\rho_{1k}^2 - \rho_{0k}^2)}{\sum_{k=1}^{n} 2G_k(\rho_{1k}^2 - \rho_{0k}^2)}\right)^{1/2} \tag{6.8}$$

$$\mathbf{v}_{0k} = (0, T)$$

Here T is an arbitrary constant. It can be seen from (6.8) that in the neighbourhood of the point (0, 0) the formula

$$\alpha = q_1^{\pm}\Omega(1 + O(\Omega)) \tag{6.9}$$

describes the origin of the first dispersion curve.

We will consider the behaviour of the dispersion curves when α and Ω approach infinity. We will assume that the limit of their ratio is a certain finite quantity, i.e. $\lim(\Omega/\alpha)$ as $\alpha \to \infty$, $\Omega \to \infty$.

Dividing (5.1) by α^2 and assuming c to be an eigenvalue parameter, we obtain a new eigenvalue problem which will be the operator analogue of the problem with a small parameter with a leading derivative. We will investigate this problem using two iterative processes of the Vishik–Lyusternik method [9, 10].

Using the first iterative process we obtain values for the first term of the asymptotic expansion of the phase velocity $c_{0k}^{(1)} = e_k$, $c_{0k}^{(2)} = \mu_k$. The second iterative process is carried out in two versions: (1) in the region of the boundary surfaces $\rho = \rho_{01}$, $\rho = \rho_{1n}$, and (2) in the region of the interfaces $\rho = \rho_{1m}$ $(m = 1, 2, \dots, n-1)$.

In the first version the first stage of the second iterative process gives the curve of the problem, which, together with this additional condition that the solution should decrease at infinity, describes a Rayleigh wave, propagating along the free surface of the cylinder with phase velocity c_s.

In the second version, for each interface $\rho = \rho_{1m}$ in the first terms of the approximation we obtain the conjugation problem, which, together with the additional condition that the solution should decay with distance from the interface, describes a Stoneley wave, propagating with phase velocity c_s.

7. As an example we will consider a three-layer cylindrical waveguide with a soft filler. In this case

$$\gamma_1^2 = c_1(d_{11}^{-1} + d_{33}^{-1}) \tag{7.1}$$

$$\mathbf{X}_1 = (X_{11}, X_{13}),\quad X_{11} = \left(\frac{d_{33}}{d_{11}}\right)^{1/2} X_0,\quad X_{13} = -\left(\frac{d_{11}}{d_{33}}\right)^{1/2} X_0,\quad X_0 = (d_{11} + d_{33})^{-1/2}$$

The first resonance frequency has the form

$$\Omega_1 = p^{1/2}\gamma_1 + O(p^{3/2}) \tag{7.2}$$

The corresponding form of the oscillations is given by formula (6.6) and (7.1).

These results enable us to conclude that in multilayer cylinders with alternating hard and soft layers at low frequencies there is more than one propagating wave, unlike a uniform cylinder of the same thickness.

We will give some results of a numerical analysis of problem (5.1) for a three-layer cylindrical waveguide. We will use Godunov's discrete orthogonalization method [11] for this purpose.

In Fig. 2 we have drawn the real part of the dispersion curves for $p = 10^{-2}$; $m_s/m_r = 0.2$; $\rho_{01} = 0.2$; $\rho_{11} = 0.5$; $\rho_{03} = 0.6$; $\rho_{13} = 1$.

The first dispersion branch, emerging from zero, as Ω increases from zero to 1.7, approximates to a straight line with a slope equal to the velocity of propagation of transverse waves in the hard layer. Beginning at $\Omega = 1.7$, the slope approaches the phase velocity of longitudinal waves in the soft layer. When α is increased further, for the first curve the asymptote will be a straight line with a slope equal to c_R.

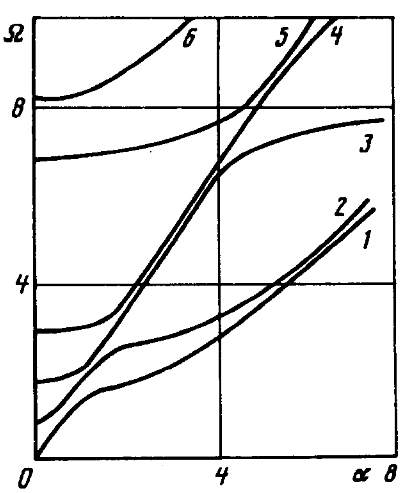

Fig. 2.

On the second dispersion branch, as Ω increases the curve approaches a straight line with a slope equal to the velocity of longitudinal waves in the hard layer. When $1.5 < \Omega < 2.8$ the slope approximates to the phase velocities of transverse waves in the hard layer.

When $\Omega > 2.8$, as Ω increases further, the slope approaches the phase velocities of longitudinal waves in the soft layer, etc. For the second curve the asymptote will be a straight line with a slope equal to $c_{0i}^{(1)}$.

Note that formula (7.2) gives the values of the first thickness resonance with an error of 7%.

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